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Local transport coefficients for chaotic systems

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Abstract. A Fokker–Planck description of chaotic dynamical systems, with phase space dependent transport coefficients is proposed. Ways of analytically obtaining these local transport coefficients are proposed. A comparison with numerical results for three model systems is made.

1. Introduction

Recently there has been a renewed interest in the study of transport properties of dynamical systems in phase space. The use of a dynamical systems approach in a number of contexts ranging from plasma physics [1] to fluid mechanics [2] make the use of a probabilistic approach to dynamical systems necessary. Since most dynamical systems of interest in applications will present chaotic behaviour, the strong local exponential instability of nearby orbits in phase space will make an orbit-by-orbit analysis of the system very hard if not impossible. Also, in most cases, it is of physical relevance to study the evolution of a collection of orbits initially distributed in a suitably defined part of phase space, as for instance would require the problem of escape of charged particles from a plasma confinement device, or the mixing of a passive tracer in a given fluid flow. In some cases the evolution of such initial probability distributions of orbits in phase space resembles fairly simple transport processes which approximately satisfy simple kinetic equations. These kinetic equations provide a particularly simple way of modelling and predicting the coarse grained features of the complicated chaotic dynamics. It is then of great importance to applications to try to characterize and quantify this transport process and find consistent ways of obtaining the transport coefficients that appear in these kinetic equations.

A number of recent interesting publications [1, 3] have proposed ways of quantifying the transport processes of dynamical systems in phase space using locally defined quantities such as the exit time from a domain. These quantities were shown to behave in a more satisfactory way than traditional measures for the transport process in the sense that they are well defined even when the transport is not diffusive in the classical sense. Such quantities have been used in the understanding of transport in cases where the motion is super-diffusive, as for instance in the case of existence of accelerator modes in the region of interest or in the case where the region of phase space where the motion takes place is bounded or shows a mixture of stochastic and regular regimes.

Most of these approaches involve the use of the notion of exit time from a domain. In [1] a local diffusion coefficient is defined analytically in terms of the exit times. However, these exit times were only obtained numerically. In this note we give a local description of the motion in phase space using analytically defined transport coefficients. We then go

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on to show how to obtain local exit times from these and also how to define a suitable averaging process which allows us to define, in certain cases, global transport coefficients.

The approach hinges on the assumption of strong stochasticity at least for the first iterations of the method proposed, and in the case that the motion can be described in terms of action angle coordinates, that the angle dynamics is randomized faster than the action dynamics. It is then possible to describe the behaviour in phase space in terms of a Fokker–Planck equation but importantly where the drift and diffusion coefficients are local functions of the action variable. The locality of the transport coefficients allows us to model a great variety of possible motions of the dynamical system under consideration. It is well known that the transport processes in typical dynamical systems can be extremely complicated, leading possibly to long correlated motions in time in certain parts of phase space (see e.g. [4, 5]) and simple kinetic equations of the Fokker–Planck-type with constant transport coefficients are incapable of showing such behaviour. However, the local approach proposed here can reproduce such effects, as shown in the examples studied in this paper.

2. The local Fokker–Planck equation

The systems we wish to study are area preserving maps of the form

$$\begin{aligned} I_{n+1} &= I_n + Kf(\theta_n) \\ \theta_{n+1} &= \theta_n + a(I_{n+1}) \end{aligned} \tag{1}$$

where $f(\theta)$ is a periodic function of the angle. For large values of K the dynamics of these maps in phase space become strongly chaotic and the angles, θ, can be assumed to evolve in a stochastic manner while the action, I, behaves in a diffusive manner. Under this framework we assume that I evolves under a stochastic differential equation (SDE)

$$\mathrm{d}I_t = -V(I_t)\,\mathrm{d}t + D(I_t)^{1/2}\,\mathrm{d}W_t \tag{2}$$

where W is a noise term which can be thought of as being a white noise term, or as a correlated noise term with a finite correlation in time, arising from the strongly chaotic (pseudo random) variation of θ. Let us quickly give a hand-waving argument why this model was chosen. We assume that for the time scales of interest, the time step of iteration of the map can be thought of as infinitesimal, an assumption which justifies the writing of the finite jump in action over this time length as a differential. We also assume that during this time the phases θ_n and $\theta_{n+\tau}$ are uncorrelated and therefore the force $Kf(\theta)$ or its iterations in time can be assumed to resemble a stochastic driving of the form $\mathrm{d}W_t$. A rigorous verification of the second assumption has, to our knowledge, never been obtained for systems other than idealized ones. However, for a great number of systems it is seen from numerical computation that this is a reasonable assumption.

Then from the general theory of stochastic equations we describe the system in terms of a Fokker–Planck equation using the Stratonovich interpretation as it automatically implies conservation of probability. The probability distribution $P(I,t)$ of the process I_t will obey the Fokker–Planck equation [6]

$$\frac{\partial P(I,t)}{\partial t} = -\frac{\partial}{\partial I}(V(I)P(I,t)) + \frac{\partial}{\partial I}\left(D(I)\frac{\partial}{\partial I}P(I,t)\right) \tag{3}$$

where $P(I,t)$ is the probability of finding the system with a value I at time t. (The Ito SDE will give a related Fokker–Planck equation and all the results given here can be trivially rewritten for this case.) The functions $V(I)$ and $D(I)$ are called the drift and diffusion coefficients respectively, and are to be interpreted as local transport coefficients.

It is important to note, that with such action-dependent functions the mean square average of I averaged over all available phase space does not necessarily change linearly with time.

These local transport coefficients can be defined as

$$V(I) = \frac{\langle I - I_0 \rangle}{\tau}$$
$$D(I) = \frac{\langle (I - I_0)^2 \rangle}{2\tau} \tag{4}$$

where the angle brackets denote averaging over the possible realizations of the noise term, and τ is the characteristic time scale over which the dynamics of the map can be thought of as being approximated by a SDE.

Our approximation procedure for obtaining analytically these local transport coefficients is based on the phase randomization assumption and can briefly be summarized as follows. As the lowest approximation we assume that the phases are randomized after a single iteration of the map. The $I - I_0$ in the definition of the transport coefficients can simply be taken to be $\Delta I_1 = I_1 - I_0$ and we assume that the angles take all possible values and thus replace the averaging over the noise by an averaging over the angles. This is nothing but the well known quasilinear approximation [7] which has been used for a long time with surprisingly good results. For the next step in the approximation we assume that one iteration of the map is not enough for the phases to become randomized and so we iterate the map twice and then take the angles every two iterations as the random variable. That is we approximate $I - I_0$ by $\Delta I_2 = I_{n+2} - I_n = F(I_n, \theta_n)$ and then perform the averaging over the 'randomized' angles θ_n. The local transport coefficients are then defined as

$$V_2(I) = -\frac{\int_{-\pi}^{\pi} \Delta I_2(I, \theta)\, \mathrm{d}\theta}{2\tau_1}$$
$$D_2(I) = \frac{\int_{-\pi}^{\pi} (\Delta I_2(I, \theta))^2\, \mathrm{d}\theta}{4\tau_1} \tag{5}$$

where τ_1 is the characteristic time for one iteration of the map and hereafter is taken to be unity, and we have dropped the subscript n. These quantities can now be interpreted as action-dependent local transport coefficients. The next approximation is obtained by using ΔI_3 instead of ΔI_2 . We will denote the local transport coefficients obtained after k iterations of the map as $V_k(I)$ and $D_k(I)$. It is easy to note that each time we go to the next approximation we recover the results of the previous approximation plus a correction term. This correction term is expected to be smaller than the previous approximation since it would consist of averaging a number of multiples of the function $f(\theta)$, which for large values of the stochasticity parameter K are expected to oscillate wildly. In all the examples studied here it is found that as one proceeds to a higher order an extra factor proportional to $K^{-1/2}$ appears at each stage and this we take as a justification of the method.

We have to emphasize once more that this approach clearly depends heavily on the validity of the phase randomization assumption. At the present moment we do not know of any rigorous results which justify this assumption for general maps of the form studied here. However, the validity of the quasilinear approximation in many cases leads us to believe that at least for strong stochasticity the assumption of randomization of the angles—at least after a number of iterates of the map, as used here—is a valid one. Also another assumption which has to be checked or can be modified is the assumption that the random phases are distributed homogeneously in the interval $[-\pi, \pi]$. There will certainly be regions of phase space where this assumption breaks down, as for instance regions near KAM curves, where the angle will not only be correlated in time for long-time intervals but will also not be

homogeneously distributed. In such cases one should try and approximate in some way the invariant distribution of angles and use this distribution in the averaging procedure that will give the local transport coefficients. The task of approximating this invariant distribution is not an easy one, but in principle could be done locally using approximation techniques on the Perron–Frobenious operator of the map, or even numerically. Since we just want to highlight certain ideas here we leave this problem for future work.

We now quote the results obtained by the method discussed above for the local transport coefficients for a model system, proposed by Antonsen and Ott [8], hereby referred to as Antonsen's hybrid map which is of the form

$$\begin{aligned} I_{n+1} &= I_n + K\sin(\theta_n) \\ \theta_{n+1} &= \theta_n + K\sin(I_{n+1}) \end{aligned} \tag{6}$$

which has been proposed in plasma physics as a model of electron heating in electromagnetic waves. The quasilinear approximation to the transport coefficients of this map gives

$$V_{QL}(I) = 0 \tag{7}$$

and

$$D_{QL}(I) = K^2/4 \tag{8}$$

which are not action dependent. The next approximation, obtained by iterating the map twice gives

$$V_2(I) = \frac{K}{2}\sum_{n=1}^{\infty} J_{2n}(K)J_1(2nK)\sin(2nI) \tag{9}$$

and

$$\begin{aligned} D_2(I) &= \frac{K^2}{4}(1 + J_0(K) + D_c(I)) \\ D_c(I) &= 2\sum_{n=1}^{\infty}[J_{2n}(K)J_0(2nK) - J_{2n}(K)J_2(2nK) - J_{2n}(2K)J_2(2nK)]\cos(2nI) \end{aligned} \tag{10}$$

where J_n is the Bessel function. Note that these transport coefficients are now action dependent but the dependence only occurs in a periodic manner. Depending on the initial position in phase space, that is the value of I, the local spreading of the trajectories will be different because of correlations of the orbits. Note that the action-independent part of the transport coefficients or equivalently the average over the action gives

$$D = \frac{K^2}{4}(1 + J_0(K)) \tag{11}$$

which is exactly the one obtained by a Fourier path method by Antonsen and Ott [8] who calculated a diffusion coefficient using the definition $D = \lim_{N\to\infty}\frac{(\Delta I)^2}{2N}$ where N is the number of iterates of the map.

As a second example we can give the standard map for which $f(\theta) = \sin(\theta)$ and $a(I) = I$. For two iterations of the map we obtain

$$\begin{aligned} V_2(I) &= -\frac{K}{2}J_1(K)\sin(I) \\ D_2(I) &= \frac{K^2}{4} + \frac{K^2}{4}(J_0(K) - J_2(K))\cos(I) - \frac{K^2}{2}J_2(2K)\cos(2I) \end{aligned} \tag{12}$$

so that the action averaged values are just these given by the quasilinear theory. If we iterate the map three times and average over angles we obtain $D_3(I) = \frac{K^2}{2}(\frac{1}{2} - J_2(K) + G(I))$ for

this map, where $G(I)$ is an oscillating function of I. The non-oscillating part is identical to that obtained by the Fourier path method by Rechester and White (see e.g. [7]). This is of course to be expected as the usual method of calculating D involves an average over all phase space.

In all the previous examples the transport coefficients could be calculated exactly in terms of known functions. For more general maps this is not always the case. To this end we give an approximate large K result for a general map of the form given in the beginning of this section. Assuming that $f(\theta)$ can be expanded in a Fourier series in the form

$$f(\theta) = \sum_{m}{}' A_m \exp(\mathrm{i}m\theta) \tag{13}$$

where the dash denotes that m never takes the value zero, we find that the twice iterated map gives

$$D_2(I) = D_{QL} + \alpha \sum_{m,m'}{}' A_m A_{m'} \int_{-\pi}^{\pi} \exp(\mathrm{i}(m+m')\theta_0) \exp(\mathrm{i}(m+m')a(I_1))\,\mathrm{d}\theta_0 \tag{14}$$

where the dash denotes that the sum is taken in such a way that $m + m' \neq 0$ and $D_{QL} = \frac{K^2}{4} \sum_m A_m A_{-m}$ and α is some numerical factor proportional to K^2. For large K the integrand $\exp(\mathrm{i}(m+m')\theta_0) \exp(\mathrm{i}(m+m')a(I_1))$ is a fast oscillating function of θ_0 and so we can use the method of stationary phase to approximate it. The dominant contribution will come from the point $\theta_0^\star$ for which

$$\frac{\mathrm{d}}{\mathrm{d}\theta_0}(\theta_0 + a(I_1)) = 0|_{\theta_0^\star} \tag{15}$$

that is

$$1 + Ka'(I_1) f'(\theta_0^\star) = 0. \tag{16}$$

For large K we can approximate this with the values of θ for which the second term vanishes thus giving two classes of points which give major contributions

$$\begin{aligned} &\theta_0^{\star,1} : f'(\theta_0^{\star,1}) = 0 \\ &\theta_0^{\star,2} : a'(I_0 + Kf(\theta_0^{\star,2})) = 0. \end{aligned} \tag{17}$$

Then a standard application of the stationary phase method (see for instance [9]) gives the dominant contribution to the diffusion coefficient

$$\begin{aligned} D_2(I) = D_{QL} + \alpha \sum_{m,m'}{}' & A_m A_{m'} ((m+m')\exp(\mathrm{i}\pi))^{-1/2} \\ & \times (\exp(\mathrm{i}(m+m')(\theta_0 + a(I_1)))(Ka'(I_1) f''(\theta_0))^{-1/2}|_1 \\ & + \exp(\mathrm{i}(m+m')(\theta_0 + a(I_1)))K^{-1}(a''(I_1)(f'^2(\theta_0)))^{-1/2}|_2) \end{aligned} \tag{18}$$

where $|_1$ and $|_2$ mean that these functions are calculated at $\theta_0^{\star,1}$ and $\theta_0^{\star,2}$ respectively. It can be seen that this diffusion coefficient will be a function of I which will not, in general, be oscillating. For instance for the Fermi map where $a(I) = I^{-1}$ the dominant contribution would contain functions of the form $I \sin(1/I)$ which are clearly not periodic. For the maps studied above the form of $a(I)$ is such that $D(I)$ is periodic in I. For the particular case of the Fermi map [12] which is a map of the form

$$\begin{aligned} &I_{n+1} - I_n + \sin(\theta_n) \\ &\theta_{n+1} = \theta_n + \frac{2\pi K}{I_{n+1}} \end{aligned} \tag{19}$$

we obtain the following asymptotic result for the diffusion coefficient

$$D_2(I) = \frac{1}{4} - \frac{1}{8}\cos\left(\frac{2\gamma}{I}\right) J_2\left(\frac{2\gamma}{I^2}\right) + \frac{1}{8}\cos\left(\frac{\gamma}{I}\right) J_0\left(\frac{\gamma}{I^2}\right) - \frac{1}{8}\cos\left(\frac{\gamma}{I}\right) J_2\left(\frac{\gamma}{I^2}\right) \tag{20}$$

where $\gamma = 2\pi K$. One can obtain similar large K results to the next approximations. We do not report these results here but note that by using the stationary-phase approximation for the integral which arises from the higher iterations of the map we always get the same stationary points as before plus some new classes of stationary points. So the next-order approximation will give the same results as the previous one plus correction terms.

Given the values of these local coefficients $V(I)$ and $D(I)$ it is relevant to understand how these quantities control the global behaviour of the system over large regions of phase space. However, before doing that it is worth noting that the method proposed here has the capability of taking into account long-time correlations of the chaotic motion. This is because we only assume randomization of the angles, after a finite number of iterations of the map. So, in a sense finite time correlations are considered in our construction of the transport coefficients. Whether the number of correlations taken into account is enough or not is a question that can be answered by looking at the convergence of the transport coefficients $D_n(I)$ and $V_n(I)$ (n denotes the number of iterations of the map taken before phase randomization was assumed) to some well-defined function $D_\infty(I)$ and $V_\infty(I)$ respectively as $n \to \infty$. The form of these local transport coefficients can also give important information on the long-time behaviour of the chaotic motion. In recent publications a number of authors have associated long-time correlated chaotic motion in dynamical systems using the concept of Levy walks. Such processes show an anomalous behaviour for some moments. In order to give a connection between these stochastic processes and the usual Fokker–Planck approach some authors (see e.g. [4] and references within) have proposed the construction of a fractional Fokker–Planck equation in action space and time which can reconstruct the probability distribution of a Levy-like process. We maintain that some of these properties (such as for instance the anomalous behaviour of the moments) can be reproduced by the use of a usual Fokker–Planck equation as long as the diffusion coefficient is allowed to be a function of action. Take as a simple example the diffusion equation, with action-dependent diffusion coefficient of the form

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial I} I^\beta \frac{\partial P}{\partial I}$$

that can have the similarity solution

$$P = t^{-\frac{1}{2-\beta}} \exp\left(-\frac{1}{(2-\beta)^2}\frac{x^{2-\beta}}{t}\right).$$

An easy calculation then shows that the moments of this process will behave as

$$\langle x^\sigma \rangle \sim t^{\frac{\sigma}{(2-\beta)}} \qquad 0 < \beta < 2$$

which reproduces a great wealth of anomalous asymptotic laws depending on the value of β. In real applications the diffusion coefficient obtained by our method can be a 'patching up' of local asymptotic behaviours with a range of values of β which will lead to a series of asymptotic-time behaviours and depending on the time regime we are interested in we will pick up a different anomalous law for the moments. Whereas in [4] the values of the fractional derivatives needed are obtained using the renormalization dynamics for the dynamical system near an island structure, here in complete analogy we obtain $D(I)$ using the dynamics (we iterate the map and then average) so the 'local' exponents, β, which govern the moment behaviour are characterized again from the dynamics. The two methods

can be thought of as being complementary as they both use knowledge of the dynamics in constructing the extension of the usual Fokker–Planck description and as they can both reproduce anomalous effects. We chose this approach as it is closer to the usual Fokker-Planck description and does not require the introduction of fractional derivatives which can be difficult to handle using conventional techniques. We would also like to draw the reader's attention to the fact that as we move to higher and higher iterations of the method and consequently more and more correlations are taken into account, the assumption of strong chaos needed for the first few iterations of the method to work may no longer be necessary as we assume randomization of the phases after the passage of longer and longer times and this can happen even under the assumption of weak chaos. An example of that is the calculation of the diffusion coefficient for the Fermi map where the diffusion coefficient (after enough iterations) seems to predict the existence of a KAM surface by the occurrence of a large dip in the region where the invariant curve is situated. Our method thus seems to be able to cope with long-time correlations if iterated sufficiently long and this is an extra indication that a local transport coefficient can describe long correlated motion, as for instance motion near an invariant curve. We conjecture that in principle one could read off the global asymptotic behaviour of the motion from local features of the transport coefficients obtained using our method to within a reasonable approximation, and we hope to provide evidence for that in a future publication. Finally we wish to note that we feel that the proposed method of constructing a systematic improvement of the 'quasilinear' transport coefficients is a useful tool since these corrections can change, in a qualitative manner, the transport process predicted by the 'quasilinear' approach and not simply in a quantitative manner. Again in this case a prime example would be the Fermi map, for which the quasilinear approach gives diffusion with a constant diffusion coefficient whereas the next corrections predict existence of a barrier to the transport in phase space.

3. Average transport coefficients

In general, because of the complicated structure of phase space the local transport coefficients are complicated functions of I. This complicated variation of the transport coefficients with action, will show if the system is studied in an orbit-like fashion, which is equivalent in the Fokker–Planck approach to studying initial probability distributions which have narrow widths in action space resembling delta functions. This, however, is not what is done in practice. From the physical point of view we need information on the evolution in phase space of a more extended initial probability distributions. For such an initial condition, we expect the intricate structure of the transport coefficients as functions of action to be smeared out as a consequence of using averages over larger regions of phase space. However, these localized effects will show on the average behaviour of extended initial conditions, leading to large scale variations in the transport coefficients. It is then important, in studying the long-time behaviour of dynamical systems, to introduce suitable averaging processes to remove the fast scale variations in phase space.

In this section we study the problem of defining average transport coefficients from the local action-dependent transport coefficients that appear in equation (3). These average transport coefficients can be compared with appropriate coefficients which can be determined from numerical computation. We give two alternative ways of the averaging process depending on how the numerical simulations are performed.

3.1. Steady state in a slab

Imagine a slab in action space whose surfaces are defined by two distinct values of the action and then set-up a constant source of particles of strength G in the middle of it. We impose absorbing boundary conditions on the ends of the slab and we look at the steady-state distribution of the particles in the slab after a number of iterations. Thus, the dynamical behaviour is determined by the following equation

$$-\frac{\partial}{\partial I}(V(I)P(I,t)) + \frac{\partial}{\partial I}(D(I)\frac{\partial}{\partial I}P(I,t)) = -G\delta(I) \tag{21}$$

with

$$P(-L/2) = P(L/2) = 0 \tag{22}$$

where L is the width of the slab. We then try to define an average diffusion and drift coefficient which will model in a satisfactory manner the steady-state distribution for this geometry. This approach has been used in the past by Cohen and Rowlands [10] for a diffusion equation and here we generalize it in the case where we include a drift coefficient.

The solution of the above equation can be written as

$$\begin{aligned} P = P_+ &= \frac{G\Lambda_2}{\Lambda_1+\Lambda_2}\frac{1}{\psi(I)}\int_I^{L/2}\frac{\psi(y)}{D(y)}\,\mathrm{d}y \qquad 0 \leqslant I \leqslant \frac{L}{2} \\ P = P_- &= \frac{G\Lambda_1}{\Lambda_1+\Lambda_2}\frac{1}{\psi(I)}\int_{-L/2}^{I}\frac{\psi(y)}{D(y)}\,\mathrm{d}y \qquad -\frac{L}{2} \leqslant I \leqslant 0 \end{aligned} \tag{23}$$

where

$$\begin{aligned} &\psi(I) = \exp\left(\int_{-L/2}^{I}\frac{V(x)}{D(x)}\,\mathrm{d}x\right) \\ &\Lambda_1 = \int_0^{L/2}\frac{\psi(x)}{D(y)}\,\mathrm{d}y \qquad \Lambda_2 = \int_{-L/2}^{0}\frac{\psi(x)}{D(y)}\,\mathrm{d}y. \end{aligned} \tag{24}$$

The total number of particles in the slab will be $N_+ + N_-$ where

$$N_+ = \int_0^{L/2}P_+(x)\,\mathrm{d}x \qquad N_- = \int_{-L/2}^{0}P_-(x)\,\mathrm{d}x. \tag{25}$$

We seek average drift and diffusion coefficients V_a and D_a which will give us the same steady-state average density. In general the result is complicated, however, here we treat the special case which is needed in the examples discussed in the paper, where we can take $V(I)$ to be an odd function of I and $D(I)$ an even function of I. Then very simple arguments show that $\Lambda_1 = \Lambda_2$ and

$$N_- = N_+ = \frac{G}{2}\int_0^{L/2}\frac{1}{\psi(y)}\int_y^{L/2}\frac{\psi(y')}{D(y')}\,\mathrm{d}y\,\mathrm{d}y'. \tag{26}$$

We want to define average values V_a and D_a in such a way that these symmetry properties are conserved. By solving the constant coefficients problem we see that the only way that $\Lambda_1 = \Lambda_2$ is if

$$V_a = 0. \tag{27}$$

Then the particle number simplifies to $N_+ = N_- = \frac{GL^2}{16D_a}$ and combining it with equations (23) and (25) we obtain

$$D_a = \frac{L^2}{8}\left[\int_0^{L/2}\frac{1}{\psi(y)}\int_y^{L/2}\frac{\psi(y')}{D(y')}\,\mathrm{d}y\,\mathrm{d}y'\right]^{-1} \tag{28}$$

which in this case where $V(I) \equiv 0$ easily reduces to the result previously given by Cohen and Rowlands [10] as $\psi(y) = 1$ for every y.

Let us now calculate the average diffusion and drift coefficient for the Antonsen map which was defined earlier. From the symmetry properties of $V(I)$ and $D(I)$ we see that $V_a = 0$. We now give a large K expansion for the effective diffusion coefficient. Since for large K, $J_n(K) \sim K^{-1/2}$ we see that

$$\begin{aligned} V &\simeq A_1 \sin(2I) + A_2 \sin(4I) + \cdots \\ D &\simeq \frac{K^2}{4}(1 + \epsilon B_1 - \epsilon^2 B_2 \cos(2I)) \end{aligned} \tag{29}$$

where $\epsilon^2 = 1/K$ and the A's and B's are oscillatory functions of K whose values are of order 1. We can now write

$$\psi(I) = \exp\left(-\epsilon^4 \int_{-L/2}^{I} 4\frac{A_1 \sin(2y) + A_2 \sin(4y) + \cdots}{1 + \epsilon B_1 - \epsilon^2 B_2 \cos(2y) + \cdots} \, \mathrm{d}y \right) = 1 - \epsilon^4 M(I) \tag{30}$$

where $M(I)$ is given by the expansion of the exponential. Since the corrections arising from this term are always going to be of order ϵ^4 and here we are interested in corrections of lower order in ϵ we can neglect the $M(I)$ contribution. We then obtain the following expansion in ϵ for the average diffusion coefficient

$$\begin{aligned} \frac{D_a}{D_{QL}} &= \frac{L^2}{8}\left(\int_0^{L/2} \int_y^{L/2} (1 - \epsilon B_1 + \epsilon^2 B_2 \cos(2y') + \cdots) \, \mathrm{d}y \, \mathrm{d}y' \right)^{-1} \\ &= \frac{L^2}{8}\left[\left(\frac{1}{8} - \frac{\epsilon B_1}{8} \right) L^2 + \frac{\epsilon^2 L \sin(L)}{4} - \frac{\epsilon^2 B_2}{4} + \frac{\epsilon^2 B_2 \cos(L)}{4} \right]^{-1} \\ &\simeq \left(\frac{1}{1 - \epsilon B_1} + \epsilon^2 \frac{\sin(L)}{4L} + \cdots \right). \end{aligned} \tag{31}$$

We see that in the limit of a large slab ($L \to \infty$)

$$\frac{D_a}{D_{QL}} \simeq \frac{1}{1 - \epsilon B_1} \simeq 1 + J_0(K) \tag{32}$$

which is the usual result obtained from the Fourier path method [8]. If the method is applied to the standard map in the limit of a large slab we obtain

$$\frac{D_a}{D_{QL}} \simeq 1 - \frac{J_2(K)}{2} + \mathrm{O}\left(\frac{1}{L} \right) \tag{33}$$

for the first term in complete accordance to the results of the Fourier path method (see e.g. [7]). So the average diffusion for a large slab only picks up the constant (non-oscillating in action) part of the local diffusion coefficient. A large slab is then equivalent to averaging over the whole of the phase space. We can also carry out the averaging over finite size slabs. Then the effective diffusion coefficient will depend on the size of the slab, as seen from equation (31), in an oscillatory manner and these oscillations are a direct consequence of the action dependence of the transport coefficients. This effective coefficient is important for modelling the transport process over finite parts of the phase space.

In the next section we give an alternative definition of an average diffusion coefficient based on a different type of numerical experiment, namely the escape of orbits from a domain.

3.2. Average transport coefficients from the asymptotic decay of an initial distribution in a slab

We now consider a numerical computation carried out in the following manner. We start some initial distribution in a slab, impose absorbing boundary conditions, follow the decay, and measure the decay rate.

The asymptotic decay rate of the probability distribution to zero is identified with the lowest non-zero eigenvalue of the eigenvalue problem

$$\begin{aligned} &-\frac{\mathrm{d}}{\mathrm{d}I}(V(I)P(I)) + \frac{\partial}{\partial I}D(I)\frac{\partial}{\partial I}P(I) = -\lambda P(I) \\ &P(-L/2) = P(L/2) = 0. \end{aligned} \tag{34}$$

We will define later an averaged diffusion coefficient in terms of this eigenvalue simply by $D_a = \frac{\lambda L^2}{\pi^2}$. There are a number of very powerful techniques for obtaining approximate values for the lowest eigenvalue, as for instance variational techniques (see e.g. [9]). However, here we shall use a perturbative method which is based on K being large, is easy to use and gives a simple final result.

For simplicity we limit ourselves to the case where the local drift coefficient $V(I)$ is smaller than the diffusion coefficient $D(I)$ as happens for the maps studied here. In accordance with the form of the transport coefficients of the Antonsen or the standard map we assume the general form

$$\begin{aligned} V(I) &= \frac{1}{\epsilon}f(I) \\ D(I) &= \frac{1}{4\epsilon^4}(1 + \epsilon b_1 + \epsilon a_1 g(I) + \cdots) \end{aligned} \tag{35}$$

where $f(I)$ and $g(I)$ are oscillating functions of I, $\epsilon = K^{-1/2}$ is a small parameter and b_1, a_1 are oscillatory functions of ϵ^{-1} with maximum value of O(1).

Defining $\bar{\lambda} = 4\epsilon^4\lambda$ we have the following eigenvalue problem

$$\begin{aligned} &4\epsilon^3\frac{\mathrm{d}}{\mathrm{d}I}(f(I)P) + \frac{\mathrm{d}}{\mathrm{d}I}(1 + \epsilon b_1 + \epsilon a_1 g(I))\frac{\mathrm{d}P}{\mathrm{d}I} = -\bar{\lambda}P \\ &P(-L/2) = P(L/2) = 0. \end{aligned} \tag{36}$$

We now use the expansion

$$\begin{aligned} P &= P_0 + \epsilon P_1 + \cdots \\ \bar{\lambda} &= \bar{\lambda}_0 + \epsilon\bar{\lambda}_1 + \cdots \end{aligned} \tag{37}$$

and get to zeroth order

$$\frac{\mathrm{d}^2P_0}{\mathrm{d}I^2} = -\bar{\lambda}_0 P_0 \qquad \text{with } P_0(-L/2) = P_0(L/2) = 0 \tag{38}$$

and to first order

$$\begin{aligned} &\frac{\mathrm{d}^2P_1}{\mathrm{d}I^2} + \bar{\lambda}_0 P_1 = -b_1\frac{\mathrm{d}^2P_0}{\mathrm{d}I^2}P_0 - a_1\frac{\partial}{\partial I}g(I)\frac{\mathrm{d}P_0}{\mathrm{d}I} - \bar{\lambda}_1 P_0 \\ &\text{with } P_1(-L/2) = P_1(L/2) = 0. \end{aligned} \tag{39}$$

The solution of the eigenvalue problem (38) gives us for the largest eigenvalue and its corresponding eigenfunction

$$P_0 = \cos\left(\frac{\pi I}{L}\right) \qquad \bar{\lambda}_0 = \frac{\pi^2}{L^2}. \tag{40}$$

Since the operator acting on P_1 on the left-hand side of (39) is self-adjoint, multiplying this equation by P_0 and integrating over the slab we obtain the consistency condition that the integral of the right-hand side of equation (39), multiplied by P_0, over the whole slab should vanish. This gives us an equation for $\bar{\lambda}_1$ in the form

$$\bar{\lambda}_1 = \left(b_1 \int_{-L/2}^{L/2} \left(\frac{\mathrm{d}P_0}{\mathrm{d}I} \right)^2 \mathrm{d}I + a_1 \int_{-L/2}^{L/2} g(I) \left(\frac{\mathrm{d}P_0}{\mathrm{d}I} \right)^2 \mathrm{d}I \right) \frac{1}{\int_{-L/2}^{L/2} P_0^2 \, \mathrm{d}I}. \tag{41}$$

Note that with the ordering given in (35) the above expression for $\bar{\lambda}_1$ is independent of the form of $V(I)$. Since in the constant diffusion coefficient case this eigenvalue is $\lambda = D\frac{\pi^2}{L^2}$, we define an average diffusion coefficient by the relation

$$D_a = \frac{L^2}{\pi^2}(\lambda_0 + \epsilon\lambda_1) \tag{42}$$

which will evidently be dependent on L, and this dependence can be interpreted as some 'average' action dependence. In the case where $g(I) = \cos(aI)$, as for instance in the Antonsen map, or the standard map, the effective diffusion coefficient to order ϵ is readily calculated to give

$$\frac{D_a}{D_{QL}} = 1 + \epsilon b_1 + \epsilon \frac{4a_1 \sin(aL/2)}{L} \frac{a^2L^2 - 2\pi^2}{a(a^2L^2 - 4\pi^2)} + \mathrm{O}(\epsilon^2) \tag{43}$$

which in the limit of large slabs $aL \gg 1$ gives the constant (in action) diffusion coefficient

$$D_a \simeq D_{QL}(1 + \epsilon b_1 + \mathrm{O}(\epsilon^2)) \tag{44}$$

which for the Antonsen map becomes

$$D_a \simeq D_{QL}(1 + J_0(K)) \tag{45}$$

and for the standard map is

$$D_a \simeq D_{QL}(1 - 2J_2(K)). \tag{46}$$

For finite slabs D_a is an oscillatory function of L and this reflects the fact that the local transport coefficients are action dependent. The local diffusion coefficient can be obtained from the average diffusion coefficient in this case by solving the integral equation

$$\int_{-L/2}^{L/2} g(I) \left(\frac{\mathrm{d}P_0}{\mathrm{d}I} \right)^2 \mathrm{d}I \frac{1}{\int_{-L/2}^{L/2} P_0^2 \, \mathrm{d}I} = \frac{D_a - D_\infty}{D_{QL}} \tag{47}$$

where $D_\infty = \lim_{L\to\infty} D_a$. Note that this way of defining the effective diffusion coefficient is consistent with the numerical procedure for the calculation of diffusion coefficients for chaotic systems proposed by Yannacopoulos and Rowlands [11]. We also note that the two different methods given in this and the previous section give equivalent results for the average diffusion coefficients.

In figure 1 we show the effective diffusion coefficient calculated numerically for the standard map for $K = 5$ by comparing the decay of an initial distribution in the slab as a function of the slab width L with that predicted by a diffusion equation with constant diffusion coefficient (see [11] for more details). The results clearly show an oscillatory dependence on L which can be very well approximated by a sinusoidal function. The amplitude of the oscillations decreases as L is increased in accordance with the results given above. The asymptotic value of D_a for large slabs tends to a value close to 5.3 which is very close to the analytically obtained value of $D_{QL}(1 - 2J_2(K))$ while the amplitude of the oscillations of D_a as a function of L around this asymptotic value is fairly well

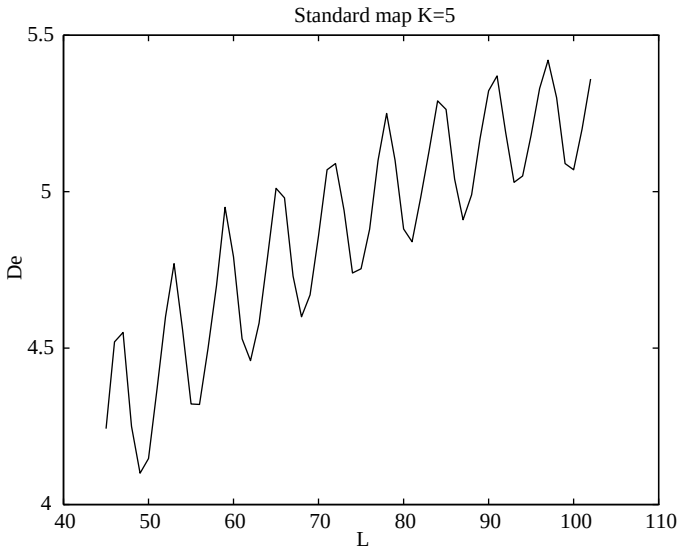

Figure 1. The average diffusion coefficient for the standard map as a function of the slab width for $K = 5$ obtained numerically using the method proposed in [11] (that is matching the escape rate for the standard map out of a rectangle of length L with that of a true diffusion process and calculating the diffusion coefficients which would give the same decay rate. Note that the important difference between this approach and the escape rate approach of [16–19] is that we do not only match the asymptotic exponential term in the series expansion for the probability distribution but a large number of terms, of the order 100 in this calculation.

predicted by an action-dependent term of the form $D_{QL}(J_0(K) - J_2(K))\cos(I)$ as given by the analytical results for the standard map (see section 2). These numerical results show that a local diffusion coefficient of the form proposed here can model, in a satisfactory manner, certain aspects of the transport in phase space†.

As a final way of testing our results we consider the Fermi map defined in section 2. Seeing how the proposed method works for this map is very important because this is a map with a bounded phase space since for large enough values of the action there exists a KAM curve which acts as a barrier to transport through phase space. The diffusion coefficients defined in section 2 cannot be calculated analytically in closed form for this map. Certain asymptotic results can be obtained such as those given in section 2. However, it is very easy to calculate the integrals defining the transport coefficients numerically as a function of I. These results are shown in figure 2. The different figures give the higher-order approximations to the diffusion coefficient as a function of the action variable, I, calculated with the proposed approximation scheme. From these results it is evident that our approximation scheme converges very fast to a local diffusion coefficient $D(I)$. We also see that the convergence is slower in the region of large I, as expected, since in the large I region the motion is ordered and the phase randomization hypothesis which is crucial to the convergence of our method is no longer valid. The small I results are unaltered when the number of iterations of the map before averaging are more than 3. However, even in the large I regime about eight iterations of the map are enough for the convergence of the method. Figure 3 shows the numerical results for the diffusion coefficient for the

† Note that in applying the averaging procedure given in this section to compare the numerically obtained transport coefficients with the analytical ones obtained in section 2 we use a slab whose ends were at $I = 0$ and $I = L$.

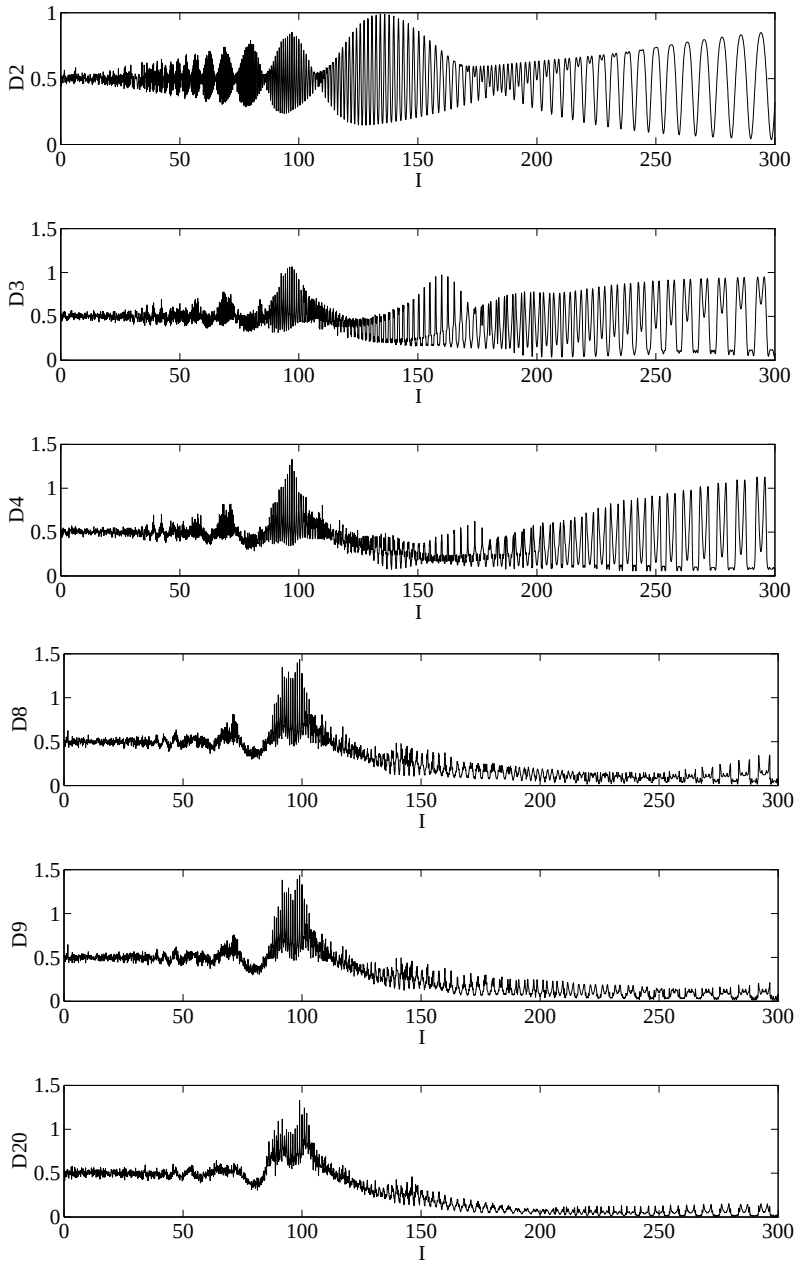

Figure 2. The diffusion coefficients obtained using the procedure proposed in section 2 for the Fermi map for a number of iterations of the map as a function of action. The number next to D shows the number of iterations used.

Fermi map obtained by Murray *et al* [12] by direct numerical simulations of the system and then fitting to a diffusion equation. Note the remarkable agreement in the form of the diffusion coefficient we showed in figure 2 for a sufficient number of iterations of the map and this numerical result. Our procedure shows the existence of a KAM torus for $I \simeq 250$

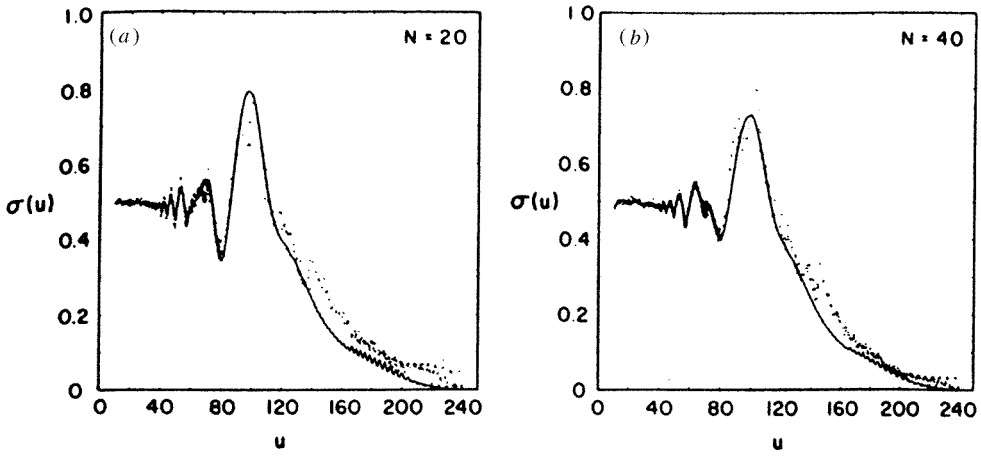

Figure 3. The diffusion coefficients obtained by numerical simulation by Murray *et al* [12] for the Fermi map. An ensemble of 64 000 particles were iterated for 20 and 40 iterations respectively and averages were taken over these orbits. Note that in this figure u stands for I. Taken from [12].

because of the great decrease of the diffusion coefficient in this region. It also reproduces the oscillations of the diffusion coefficient in action space and predicts well the position of the maximum as well as the value of the diffusion coefficient there. In figure 4 we show results for the same map obtained in [11] using the decay of the probability distribution in a slab and again we note a very good resemblance. (Note that figure 4 gives a diffusion coefficient which is half the one in figure 3, this is due to the definition of the diffusion coefficient used by Murray *et al* [12].) Figure 4 does not show the oscillations shown in figure 3, this is because of the numerical method used which gives an average diffusion over regions of action space thus removing the fast oscillations. For more details concerning this see the discussion in [11].

From figure 2 we note that the deviation of the transport coefficients from the quasilinear value in some regions of action space is of almost the same order of magnitude as the quasilinear value itself. For these values of the action the diffusion coefficient cannot be considered as a perturbation about a constant transport coefficient the method for defining the average diffusion coefficient given in this section does not strictly apply. Such an approach gives interesting results far away from the KAM curve but is seen to break down as expected near the KAM curve. However, from the same figure we note that the local diffusion coefficients obtained by the method given in section 2 display oscillations in two distinct action scales. There is a slow oscillation very similar to the one obtained numerically by Murray *et al* [12] and displayed in figure 3, which constitutes the envelope of the action varying diffusion coefficient and a fast oscillation on top of this slow variation. It is evident that the system is unable to respond to this fast varying action variation for initial orbit distribution which are not of a singular (delta function in action) type. From standard homogenization arguments using multiple scales perturbation theory (in a way very similar to the one used in [10], but leaving the 'slow' action dependence in the reduced equations) we can find that the slow evolution of the probability distribution will be given

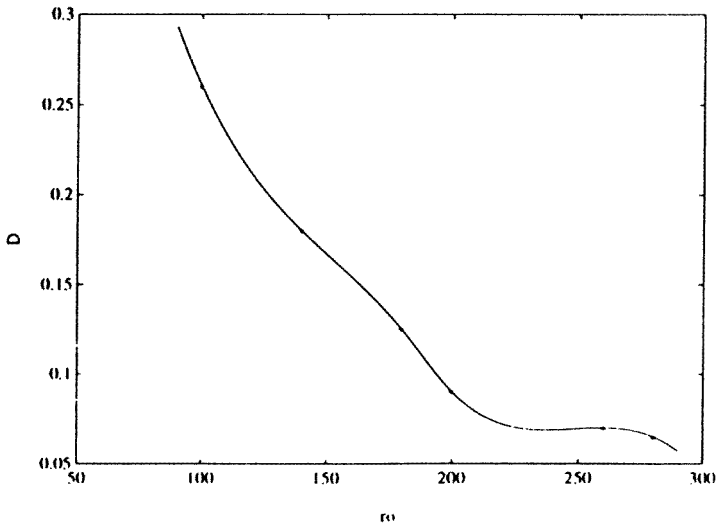

Figure 4. The diffusion coefficient for the Fermi map obtained by Yannacopoulos and Rowlands in [11]. Note that in this figure r_0 stands for L. Taken from [11].

by a diffusion equation with an 'effective' (homogenized) diffusion coefficient of the form

$$D(I_0) = \left(\int_Q \frac{1}{D(I_1, I_0)} \, \mathrm{d}I_1 \right)^{-1} \tag{48}$$

where I_0 and I_1 denote the slow and fast action scales respectively and Q is the periodicity cell in the fast variable. For a proof of this result see the appendix. The leading-order approximation to this homogenized diffusion coefficient would simply be the average of the full diffusion coefficient over the fast scale. From the asymptotic expansion of D_2 in section 2 for the standard map we observe that the diffusion coefficient contains terms of the form $\cos(\frac{\gamma}{I}) J_n(\frac{\gamma}{I^2})$ for various integers n. The fast oscillations in the diffusion coefficient are due to the trigonometric terms while the slow oscillation is due to the Bessel function terms. The fast variable can then be associated to $I_1 = \frac{\gamma}{I}$ and the slow variable to $I_0 = \frac{\gamma}{I^2}$. The average of the diffusion coefficient over the fast variable in the original action variable can then be rewritten as

$$D_a(I) = \frac{\gamma}{2\pi} \int_{I_e}^{I} \frac{D(I_1, I_0)}{I'^2} \, \mathrm{d}I' \tag{49}$$

where I_1 and I_0 are functions of I as defined above and

$$I_e = \frac{\gamma I}{\gamma + 2\pi I}. \tag{50}$$

The new limits of integration arise from the observation that the diffusion coefficient is a periodic function in the fast variable I_1 so that the averaging over a periodicity cell Q in this variable is an integration between $I_1 = I_{10}$ and $I_1 = I_{10} + 2\pi$. This homogenized diffusion coefficient as a function of the action was calculated numerically using the diffusion coefficients obtained from the procedure of section 2. In figure 5 the results obtained by this method by iterating the map four times and nine times are shown. The homogenized diffusion coefficients show remarkable agreement with the numerically observed ones by Murray *et al* [12] reproduced in figure 3 and support the argument that the system will not

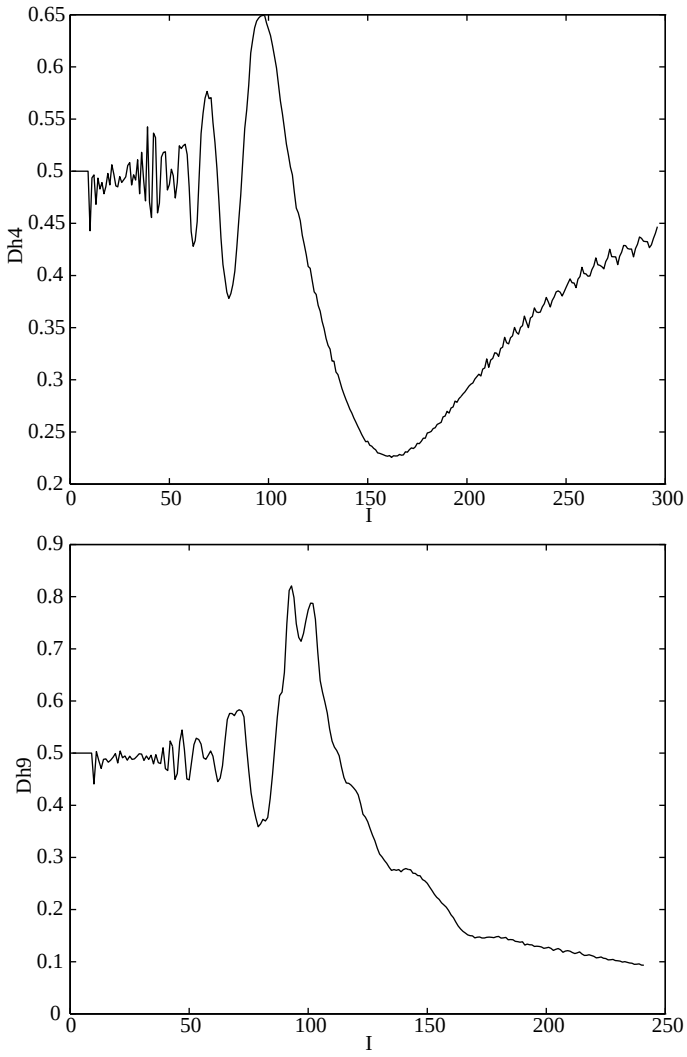

Figure 5. The 'homogenized' (average) diffusion coefficient (as defined by equation (48)) after averaging short action scale effects obtained theoretically using the four times and nine times iterated map.

respond to the fast oscillations in action space for reasonably extended initial probability distributions but will follow the envelope of the diffusion coefficient. The fast oscillations will be relevant in the case where the initial probability distribution is a delta function or a very localized function in action space.

In closing this section let us briefly draw analogy of our method with some recent results obtained using the escape rate formulation [13–16]. First, defining the averaged diffusion coefficient through the escape rate is equivalent in defining the average diffusion coefficient by the first eigenvalue of the associated Dirichlet problem, since an asymptotic expansion of the solution will show that the particle number in the domain will behave asymptotically in time as $N(t) \sim \exp(-\lambda_1 t)$. What is of more interest is the recent association of the transport coefficients with dynamical quantities such as the Lyapunov exponents and the Hausdorff

dimension of the subset of phase space which consists of all the (unstable) periodic orbits of the dynamical system which was proposed by Gaspard and Nicolis [13] (see also [14] for a more detailed discussion). More precisely they find that the average diffusion coefficient defined by the escape rate will be proportional to the average Lyapunov exponent on this set multiplied by the Hausdorf codimension of this set ($c_H = 1 - d_H$). We feel this can be reproduced within our proposed method. This set of recurrent orbits will show in our calculation of $D_\infty(I)$ as a series of large dips, since for values of I on this set $I_n - I$ will be small. However, the depth of these dips will be a function of the Lyapunov exponent. As $D_\infty(I)$ is obtained by averaging over all angles and the periodic orbits in the recurrent set are defined by both an angle and an action coordinate, the value we will obtain for the transport coefficient will depend on the behaviour of nearby trajectories i.e. trajectories with the same value for the action but different value for the angles. This is controlled by the Lyapunov exponent. On the basis of this simple argument we expect the dips in the diffusion coefficient obtained with our method to get smaller as the Lyapunov exponent gets larger. We then end up with a fast oscillating function $D(I)$ which has a number of dips whose size is controlled by the Lyapunov exponent. In defining the average diffusion coefficient by the exit rate or equivalently by the first non-zero eigenvalue of the associated Dirichlet problem we can intuitively understand that this decay rate is going to depend not only in the size of the dips but also on how often these dips occur. The more often these dips the smaller the decay rate. This is controlled by the Hausdorff dimension of the recurrent set. In this way we see that one can reproduce, at least qualitatively, the connection between the effective transport coefficients and the Lyapunov exponents and the Hausdorff dimension of the recurrent set in the context of the present method. A rigorous mathematical proof of this statement is currently under investigation, but as it is beyond the scope of this paper we hope to report on it in a future publication.

We consider that of the two methods of defining effective diffusion coefficients proposed here, the second one is preferable since it can easily be generalized to higher-dimensional maps.

4. Exit times

In this section we give estimates for the first exit times from a domain using the local transport coefficients obtained here and show that using these results we can approximate, analytically, certain quantities defined by Benkadda *et al* [3] and which were shown numerically to give very good indicators of transport.

The average first exit time $\sigma(I)$ from a domain as a function of the initial position in phase space is given by the solution of the following boundary value problem [17]

$$\begin{aligned} &(V(I)\sigma(I))' + (D(I)\sigma(I)')' = -1 \\ &\sigma(I_0 - L/2) = \sigma(I_0 + L/2) = 0 \end{aligned} \tag{51}$$

where the dash denotes differentiation with respect to I and the domain chosen is a slab centred at I_0 with width L. The solution to this problem can be readily written as

$$\begin{aligned} \sigma(I) &= \frac{1}{\psi(I)} \int_{I_0 - L/2}^{I} \frac{-y + C}{D(y)} \psi(y)\, \mathrm{d}y \\ \psi(I) &= \exp\left(\int_{I_0 - L/2}^{I} \frac{V(y)}{D(y)}\, \mathrm{d}y \right) \end{aligned} \tag{52}$$

where C is a constant given by

$$C = \left(\int_{I_0-L/2}^{I_0+L/2} \frac{y\psi(y)}{D(y)}\,\mathrm{d}y \right) \left[\int_{I_0-L/2}^{I_0+L/2} \frac{\psi(y)}{D(y)}\,\mathrm{d}y \right]^{-1}. \tag{53}$$

For the maps of the form used here $V(I)$ is of several orders of magnitude smaller that $D(I)$ at least in the large K limit and so to the order of approximation that we are interested in we can take $\psi(I) \simeq 1$. Then assuming that $D(I) = \frac{K^2}{4}(A_1 + \epsilon A_2 f_1(I) + \epsilon^2 A_2 f_2(I) + \ldots)$ and expanding as in the last section we obtain to order ϵ

$$\sigma(I) = \frac{L^2}{8A_1} + \epsilon \frac{\Gamma_1 L}{2A_1} + \epsilon \frac{A_2 \Pi_1}{A_1^2} - \epsilon I_0 A_2 \Pi_2 A_1^2 + \mathrm{O}(\epsilon^2) \tag{54}$$

where Γ_1 and Π_1, Π_2 are given as integrals of $f_1(I)$ and $A_1 = 1 + J_0(K)$ for the Antonsen map or $A_1 = 1 - 2J_2(K)$ for the standard map. Rather than give the general expressions here, which are lengthy, but straightforward to obtain, we just give the result for the special case where $f_1(I) = \cos(mI)$ namely,

$$\sigma(I, I_0) = \frac{L^2}{8A_1} - \epsilon \frac{A_2 \cos(mI_0)\cos(mL/2)}{m^2 A_1^2} - \frac{\epsilon}{2} \frac{A_2 L \cos(mI_0)\sin(mL/2)}{mA_1^2} + \epsilon \frac{A_2 \cos(mI_0)}{m^2 A_1^2} \tag{55}$$

where we have added the variable I_0 in the exit time to characterize the centre of the slab we are working with.

In their paper Benkadda *et al* [3] introduced a measure for the study of transport, which quoting from their paper, can be defined as follows. Given an initial point I_0 at $t_0 = 0$ they choose a first domain Ω_0 containing I_0 and then determine the exit time $\tau(I_0, \Omega_0)$, then associate a new position I_{t_1} outside the initial domain, and a new domain Ω_1 and find the exit time $\tau(I_{t_1}, \Omega_1)$. This procedure is repeated to obtain an orbit average $\langle T \rangle(I_0)$ of exit times (if it exists)

$$\langle T \rangle(I_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \tau(I_{t_k}, \Omega_k). \tag{56}$$

Their numerical results showed that this measure can describe 'global' aspects of transport such as an effective (averaged) diffusion coefficient D_a and in fact their numerics show that this quantity behaves in accordance with $\frac{1}{D}$.

Here, we show how to obtain an analytical estimate for this quantity using the exit times obtained from the local transport coefficients proposed here. In the formula defining $\langle T \rangle$ we substitute τ with the mean exit times $\sigma(I, I_0)$ and we change the orbit averaging with a phase-space average, by taking a sequence of identical slabs centred at $I_i = I_0 + \mathrm{i}\Delta I$ and then averaging over a large number of them. This gives

$$\langle T \rangle(I_0) = \frac{L^2}{8A_1} + \epsilon F \frac{\cos(m\Delta I N)}{N} + \epsilon G \frac{\sin(m\Delta I N)}{N} + \cdots \tag{57}$$

where

$$F = \left(-\frac{A_2 \cos(mI_0)\sin(mL/2)}{m^2 A_1^2} - \frac{A_2 L \cos(mI_0)\sin(mL/2)}{2mA_1^2} + \frac{A_2\cos(mI_0)}{m^2A_1^2} \right) \times \frac{\sin(m\Delta I + mI_0) + \sin(mI_0)}{2\sin(m\Delta I)}$$
$$G = \left(-\frac{A_2 \cos(mI_0)\sin(mL/2)}{m^2 A_1^2} - \frac{A_2 L \cos(mI_0)\sin(mL/2)}{2mA_1^2} + \frac{A_2\cos(mI_0)}{m^2A_1^2} \right) \times \frac{\cos(m\Delta I + mI_0) + \cos(mI_0)}{2\sin(m\Delta I)}. \tag{58}$$

We now note that in the limit as $N \to \infty$ and taking into account the results of section 3,

$$\langle T \rangle \to \frac{L^2}{8A_1} \propto \frac{1}{D_a} \tag{59}$$

in accordance with the numerical results of Benkadda *et al* [3]. However, for finite N there will also be an action dependence in this quantity reflecting the fact that it still keeps some local information.

Closing this section we would like to make a connection with the work of Zumofen and Klafter [19] that associate anomalous diffusion in the standard map with the exit time distribution. Using our method and the results of this section we can obtain an analytical average first exit time distribution. As the minima in the diffusion coefficient obtained will correspond to large spikes in the exit time distribution our results for the exit time will reproduce some of the features for the exit time plotted in figure 3 of [19], or rather the average of this exit time over the possible phases. In this way one could approximate using analytical techniques some of the features of the exit time distribution which is obtained numerically in [19]. It can be also seen that an action-dependent diffusion coefficient can reproduce power laws in the exit time distribution. For instance, a diffusion coefficient of the form $D(I) \sim I^\beta$ will give average exit times of the form $\sigma(I) \sim c_1 I^{1-\beta} + c_2 I^{2-\beta}$ which in turn reflect a power law exit time distribution similar to the one proposed in [19] using Levy walk arguments. The problem of relating exit time distributions for anomalous diffusion using a diffusion equation with a power law in action diffusion coefficient has also been studied by Pikovski [20]. Furthermore, in certain cases where our method for defining the diffusion coefficient shows poor convergence as a function of the iterations used (because of the existence of long-time correlations, e.g. the case of accelerator modes in the standard map) we can put an explicit time dependence in the analytically calculated diffusion coefficient leading to an even greater wealth of possible asymptotic laws resembling more and more long correlated Levy walks.

5. Generalization to higher-dimensional maps

The results obtained in this paper are readily generalized to higher-dimensional maps which are generalizations of the two-dimensional maps introduced in section 2. Now an averaging over the different angles must be made and instead of scalar drift and diffusion coefficients we will have a vector drift coefficient $(V_1(\boldsymbol{I}), V_2(\boldsymbol{I}), \ldots)$ and a diffusion tensor $D_{ij}(\boldsymbol{I})$ where $i, j = 1 \ldots n$, n the number of the actions, and $\boldsymbol{I} = (I_1, \ldots, I_n)$. The diffusion equation will then be

$$\frac{\partial P(\boldsymbol{I}, t)}{\partial t} = \sum_i \frac{\partial V_i(\boldsymbol{I}) P(\boldsymbol{I}, t)}{\partial I_i} + \sum_{ij} \frac{\partial}{\partial I_i} D_{ij}(\boldsymbol{I}) \frac{\partial P(\boldsymbol{I}, t)}{\partial I_j}. \tag{60}$$

An average diffusion coefficient can now be obtained using the decay rate of an initial probability distribution in a slab with absorbing boundary conditions. While some very efficient variational techniques can be used in the higher-dimensional case as well and which can give bounds on the effective diffusion and drift coefficients in this case we prefer to use a perturbative technique in more or less the spirit of the one used in section 3.2. We work with the special case of coupled standard-like maps of the form

$$\begin{aligned} I_{1,n+1} &= I_1 + F_1(\theta_{1,n}, \theta_{2,n}) \\ I_{2,n+1} &= I_2 + F_2(\theta_{1,n}, \theta_{2,n}) \\ \theta_{1,n+1} &= \theta_{1,n} + \omega_1(I_{1,n+1}, I_{2,n+1}) \\ \theta_{2,n+1} &= \theta_{2,n} + \omega_2(I_{1,n+1}, I_{2,n+1}) \end{aligned} \tag{61}$$

for which the local drift vector and local diffusion tensor will be of the form

$$\begin{aligned} V_1 &= \frac{1}{\epsilon} f_1(I_1, I_2), \quad V_2 = \frac{1}{\epsilon} f_2(I_1, I_2) \\ D_{11} &= \frac{1}{4\epsilon^4}(A_1 + \epsilon g_1(I_1, I_2) + \cdots) \\ D_{22} &= \frac{1}{4\epsilon^4}(A_2 + \epsilon g_2(I_1, I_2) + \cdots) \\ D_{12} &= \mathrm{O}(\epsilon) \end{aligned} \tag{62}$$

where $\epsilon^2 = 1/K$. Note that this specific form of the transport coefficients is not necessary for the application of the method.

We now have to solve the eigenvalue problem $\hat{L}_{FP} P(\boldsymbol{I}) = -\lambda P(\boldsymbol{I})$ where $\hat{L}_{FP}$ is the spatial part of the Fokker–Planck equation. The boundary conditions imposed are absorbing boundary conditions on a higher-dimensional slab. Rescaling the eigenvalue by defining $\bar{\lambda} = 4\epsilon^4\lambda$ and using the expansion

$$\begin{aligned} P &= P_0 + \epsilon P_1 + \cdots \\ \bar{\lambda} &= \bar{\lambda}_0 + \epsilon\bar{\lambda}_1 + \cdots \end{aligned} \tag{63}$$

we separate orders of ϵ to obtain to zeroth order

$$\begin{aligned} &\hat{L}_0 P_0 = \left(A_1 \frac{\partial^2}{\partial I_1^2} + A_2 \frac{\partial^2}{\partial I_2^2}\right) P_0 = -\bar{\lambda}_0 P_0 \\ &P_0(-L_1/2, I_2) = P_0(L_1/2, I_2) = 0 \qquad P_0(I_1, -L_2/2) = P_0(I_1, L_2/2) = 0 \end{aligned} \tag{64}$$

and to first order

$$\begin{aligned} &\hat{L}_0 P_1 = -\bar{\lambda}_0 P_1 - \bar{\lambda}_1 P_0 + \frac{\partial}{\partial I_1} g_1(I_1, I_2) \frac{\partial}{\partial I_1} P_0 + \frac{\partial}{\partial I_2} g_2(I_1, I_2) \frac{\partial}{\partial I_2} P_0 \\ &P_1(-L_1/2, I_2) = P_1(L_1/2, I_2) = 0 \qquad P_1(I_1, -L_2/2) = P_1(I_1, L_2/2) = 0. \end{aligned} \tag{65}$$

The zeroth-order problem can be solved to give the eigenfunctions

$$\begin{aligned} P_{0nm} &= \cos\left(\frac{\pi n I_1}{L_1}\right) \cos\left(\frac{\pi m I_2}{L_2}\right) \qquad n, m odd \\ \bar{\lambda}_{0nm} &= A_1 \frac{n^2\pi}{L_1^2} + A_2 \frac{m^2\pi}{L_2^2}. \end{aligned} \tag{66}$$

Using the self-adjointness of the operator $\hat{L}_0$ we can find $\bar{\lambda}_{1nm}$ from a consistency condition similar to the one used for the one-dimensional case. This yields

$$\bar{\lambda}_{1nm} = \left(\int_{-L_1/2}^{L_1/2} \int_{-L_2/2}^{L_2/2} \left(\frac{\partial}{\partial I_1} g_1(I_1, I_2) \frac{\partial}{\partial I_1} P_{0nm} + \frac{\partial}{\partial I_2} g_2(I_1, I_2) \frac{\partial}{\partial I_2} P_{0nm} \right) P_{0nm} \, \mathrm{d}I_1 \, \mathrm{d}I_2 \right) \times \left(\int_{-L_1/2}^{L_1/2} \int_{-L_2/2}^{L_2/2} P_{0nm}^2 \, \mathrm{d}I_1 \, \mathrm{d}I_2 \right)^{-1}. \tag{67}$$

Matching this decay rate with the one obtained from the constant drift and diffusion case we can define an effective diffusion tensor which would model adequately the process, as far as the escape of particles from a given domain is concerned.

6. Concluding remarks

In this short note we have proposed a way of obtaining local transport coefficient for chaotic systems. The method is a formal one which is based on the assumption of randomization of the angle coordinates after a sufficiently high number of iterations. The first approximation is nothing but the well known quasilinear approximation which takes the angles as totally uncorrelated after one iteration of the map. The higher-order approximations are obtained by assuming that angle randomization occurs after two or more iterations of the map. This extension alleviates the need for strong chaos for the validity of the Fokker–Planck description and leads to the possibility of introducing long-time correlations in the motion. Furthermore, allowing the transport coefficients to be functions of action we can reproduce to some extent results on the transport in phase space that have been modelled with other techniques such as use of Levy walk models or transport equations containing fractional derivatives. Some ways of getting effective diffusion coefficients from these local ones have been given. Connections with previously used techniques for obtaining transport coefficients have also been made. The extremely interesting problem which is left open is to put on a rigorous basis these angle randomization assumptions, which to the best of our knowledge has not yet been dealt successfully except in the case of certain simple maps [18].

Appendix: Derivation of equation (48)

In this appendix we highlight the derivation of equation (48) concerning the homogenized (or average) diffusion coefficient in the case where the local diffusion coefficient depends on a fast and slow scale in the action. Assume a diffusion equation of the form

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial I} D\left(I, \frac{I}{\epsilon}\right) \frac{\partial P}{\partial I} \tag{68}$$

and define $I_1 = \epsilon^{-1} I_0$, $I_0 = I$ and $t_0 = t$, $t_1 = \epsilon^{-1} t0$ and $t_2 = \epsilon^{-2} t_0$. Expanding in these multiple scales and using the ansatz $P = P_0 + \epsilon P_1 + \ldots$ for the probability distribution we obtain to the first orders

$$\begin{aligned} &\frac{\partial}{\partial I_1} D \frac{\partial P_0}{\partial I_1} = \frac{\partial P_0}{\partial t_2} \\ &\frac{\partial}{\partial I_0} D \frac{\partial P_0}{\partial I_1} + \frac{\partial}{\partial I_1} D \frac{\partial P_0}{\partial I_0} + \frac{\partial}{\partial I_1} D \frac{\partial P_1}{\partial I_1} = \frac{\partial P_1}{\partial t_2} + \frac{\partial P_0}{\partial t_1} \\ &\frac{\partial}{\partial I_0} D \frac{\partial P_0}{\partial I_0} + \frac{\partial}{\partial I_0} D \frac{\partial P_1}{\partial I_1} + \frac{\partial}{\partial I_1} D \frac{\partial P_1}{\partial I_0} + \frac{\partial}{\partial I_1} D \frac{\partial P_2}{\partial I_1} = \frac{\partial P_0}{\partial t_0} + \frac{\partial P_1}{\partial t_1} + \frac{\partial P_2}{\partial t_2}. \end{aligned} \tag{69}$$

Averaging the first equation over t_2 and imposing the periodicity condition on I_1 we find that $\bar{P}_0$ is only dependent on I_0, where the overbar denotes averaging over t_2. Then averaging the second equation over t_1 and t_2 we obtain that

$$D\left(\frac{\partial \bar{\bar{P}}_1}{\partial I_1}+\frac{\partial \bar{\bar{P}}_0}{\partial I_0}\right)=C(I_0) \tag{70}$$

where the double overbars now denote averaging over both t_1 and t_2. Then the next-order equation gives, after averaging over both t_1 and t_2 and imposing periodicity in I_1, the consistency condition

$$\frac{\partial}{\partial I_0}\int_Q D\,\mathrm{d}I_1\frac{\partial \bar{\bar{P}}_0}{\partial I_0}+\frac{\partial}{\partial I_0}\int_Q D\frac{\partial \bar{\bar{P}}_1}{\partial I_1}\,\mathrm{d}I_1=\frac{\partial \bar{\bar{P}}_0}{\partial t_0}. \tag{71}$$

We then assume that $\bar{\bar{P}}_1$ can be written in the form

$$\bar{\bar{P}}_1=g(I_1)\frac{\partial \bar{\bar{P}}_0}{\partial I_0}+\tilde{P}_0(I_0)$$

with g periodic in I_1. Then g solves the equation

$$D\frac{\mathrm{d}g}{\mathrm{d}I_1}=-D+C(I_0)$$

which is consistent with periodicity of g if

$$C(I_0)^{-1}=\int_Q D^{-1}\,\mathrm{d}I_1\equiv D_a. \tag{72}$$

With this choice for $\bar{\bar{P}}_1$ one can then clearly see that $\bar{\bar{P}}_0$ satisfies the diffusion equation

$$\frac{\partial \bar{\bar{P}}_0}{\partial t_0}=\frac{\partial}{\partial I_0}D_a\frac{\partial \bar{\bar{P}}_0}{\partial I_0} \tag{73}$$

which is equivalent to equation (48).

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